

‘Hidden’ Symmetries of Higher Dimensional Rotating Black Holes

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We demonstrate that the rotating black holes in an arbitrary number of dimensions and without any restrictions on their rotation parameters possess the same ‘hidden’ symmetry as the 4-dimensional Kerr metric. Namely, besides the spacetime symmetries generated by the Killing vectors they also admit the (antisymmetric) Killing–Yano and symmetric Killing tensors.

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The idea that the spacetime may have one or more large spatial extra dimensions became very popular recently. In the brane world models, which realize this idea, the usual matter is confined to the brane, representing our world, while gravity propagates in the bulk. Black holes, being the gravitational solitons, propagate in the bulk and may be used as probes of extra dimensions. If the size r_0 of a black hole is much smaller than the size L of extra dimensions and the black-hole–brane interaction is weak, the black hole geometry is distorted only slightly. This distortion is controlled by the dimensionless parameter r_0/L . For many problems it is sufficient to consider the limit when this parameter vanishes and approximate the geometry by the metric of an isolated black hole. The metrics describing the isolated vacuum rotating higher dimensional black holes were obtained by Myers and Perry [1]. These solutions are the generalizations of the well known 4-dimensional Kerr geometry. The symmetries play a key role in the study of physical effects in the gravitational fields of black holes. In this paper we demonstrate that the Myers–Perry metrics besides the evident symmetries possess also an additional ‘hidden’ symmetry in the same way as it occurs for the Kerr spacetime.

We start by reminding that the Kerr metric possesses a number of what was called by Chandrasekhar [2] ‘miraculous’ properties. This metric was obtained by Kerr [3] as a special solution which can be presented in the Kerr–Schild form

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Hl_\mu l_\nu, \quad (1)$$

where $\eta_{\mu\nu}$ is a flat metric and l_μ is a null vector, in both metrics g and η . The Kerr solution is stationary and axisymmetric, and it belongs to the metrics of the special algebraic type D. Although the Killing vector fields ∂_t and ∂_ϕ are not enough to provide a sufficient number of integrals of motion, Carter [4] demonstrated that both—the Hamilton–Jacobi and scalar field equations—can be separated in the Kerr metric. This ‘miracle’ is directly connected with the existence of an additional integral of

motion associated with the second rank Killing tensor [5] $K_{\mu\nu} = K_{(\mu\nu)}$ obeying the equation

$$K_{(\mu\nu;\lambda)} = 0. \quad (2)$$

As it was shown later, the equations for massless fields with non-vanishing spin can be decoupled in this background, and the variables separated in the resulting Teukolsky’s master equations [6, 7].

Penrose and Floyd [8] demonstrated that the Killing tensor for the Kerr metric can be written in the form $K_{\mu\nu} = f_{\mu\alpha} f_\nu{}^\alpha$, where the antisymmetric tensor $f_{\mu\nu} = f_{[\mu\nu]}$ is the Killing–Yano (KY) tensor [9] obeying the equation $f_{\mu(\nu;\lambda)} = 0$. Using this object, Carter and McLenaghan [10], constructed the symmetry operator of the massive Dirac equation.

In many aspects a KY tensor is more fundamental than a Killing tensor. Namely, its ‘square’ is always Killing tensor, but the opposite is not generally true (see, e.g., [11]). In 4-dimensional spacetime, as it was shown by Collinson [12], if a vacuum solution of the Einstein equations allows a non-degenerate KY tensor it is of the type D. All the vacuum type D solutions were obtained by Kinnersley [13]. Demianski and Francaviglia [14] showed that in the absence of the acceleration these solutions admit Killing and KY tensors. It should be also mentioned that if a spacetime admits a non-degenerate KY tensor it always has at least one Killing vector [15].

One can expect that at least some of these deep relations between ‘hidden’ symmetries and the algebraical structure of solutions of the Einstein equations remain valid also in higher dimensional spacetimes.

Really, it was demonstrated that the 5-dimensional rotating black hole metric possesses the Killing tensor and allows the separation of variables of the Hamilton–Jacobi and scalar field equations [16, 17]. This separation is also possible in higher dimensional rotating black hole metrics under a condition that their rotation parameters can be divided into two classes, and within each of the classes the rotation parameters are equal one to another [18]. Below we show that all the known vacuum rotating black hole metrics in an arbitrary number of dimensions and without any restrictions on their rotation parameters admit both the Killing–Yano and the Killing tensors.

The Myers–Perry (MP) metrics [1] are the most general known vacuum solutions for the higher dimensional

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rotating black holes [19]. These metrics allow the Kerr–Schild form (1), and, as it was shown recently [21], they are of the type D. The MP solutions have slightly different form for the odd and even number of spacetime dimensions D . We can write them compactly as

$$ds^2 = -dt^2 + \frac{U dr^2}{V - 2M} + \frac{2M}{U} \left(dt + \sum_{i=1}^n a_i \mu_i^2 d\phi_i \right)^2 + \sum_{i=1}^n (r^2 + a_i^2) (\mu_i^2 d\phi_i^2 + d\mu_i^2) + \varepsilon r^2 d\mu_{n+\varepsilon}^2, \quad (3)$$

where

$$V = r^{\varepsilon-2} \prod_{i=1}^n (r^2 + a_i^2), \quad U = V \left(1 - \sum_{i=1}^n \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \right). \quad (4)$$

Here $n = [(D-1)/2]$, where $[A]$ means the integer part of A . We define ε to be 1 for D even and 0 for odd. The coordinates μ_i are not independent. They obey the following constraint

$$\sum_{i=1}^n \mu_i^2 + \varepsilon \mu_{n+\varepsilon}^2 = 1. \quad (5)$$

The MP metrics possess $n+1$ Killing vectors, $\partial_t, \partial_{\phi_i}, i = 1, \dots, n$.

We show now that there is an additional symmetry connected with the KY tensors. The KY tensor is a special case of what is called a *conformal KY tensor* which is defined as a p -form k obeying the equation [22, 23, 24]

$$\nabla_{(\alpha_1} k_{\alpha_2) \dots \alpha_{p+1}} = g_{\alpha_1 \alpha_2} \Phi_{\alpha_3 \dots \alpha_{p+1}} - (p-1) g_{[\alpha_3 (\alpha_1} \Phi_{\alpha_2) \alpha_4 \dots \alpha_{p+1}]}, \quad (6)$$

$$\Phi_{\alpha_3 \alpha_4 \dots \alpha_{p+1}} = \frac{1}{D+1-p} \nabla_{\beta} k^{\beta}_{\alpha_3 \alpha_4 \dots \alpha_{p+1}}. \quad (7)$$

KY tensors themselves form a subset of all conformal KY tensors for which $\Phi = 0$. Thus the KY tensor of rank p is a p -form $f_{\alpha_1 \dots \alpha_p}$ obeying the equation

$$\nabla_{(\alpha_1} f_{\alpha_2) \alpha_3 \dots \alpha_{p+1}} = 0. \quad (8)$$

In what follows we shall use the following properties. Denote by $e_{\alpha_1 \dots \alpha_D}$ the totally antisymmetric tensor

$$e_{\alpha_1 \dots \alpha_D} = \sqrt{-g} \epsilon_{\alpha_1 \dots \alpha_D}, \quad e^{\alpha_1 \dots \alpha_D} = -\frac{1}{\sqrt{-g}} \epsilon^{\alpha_1 \dots \alpha_D}. \quad (9)$$

This tensor obeys the property ($q+p=D$)

$$e_{\mu_1 \dots \mu_q \beta_1 \dots \beta_p} e^{\nu_1 \dots \nu_q \beta_1 \dots \beta_p} = -p! q! \delta_{\mu_1}^{[\nu_1} \dots \delta_{\mu_q}^{\nu_q]}. \quad (10)$$

The Hodge dual $*\omega$ of the form ω is defined as

$$(*\omega)_{\mu_1 \dots \mu_{D-p}} = \frac{1}{p!} e_{\mu_1 \dots \mu_{D-p}}^{\alpha_1 \dots \alpha_p} \omega_{\alpha_1 \dots \alpha_p}. \quad (11)$$

One can check that $*(*\omega) = -\omega$. The Hodge dual of a conformal KY tensor is again a conformal KY tensor. A

conformal KY tensor k is dual to the KY tensor if and only if it is closed $dk = 0$ [24].

We focus our attention on the KY tensor f of the rank $p = D-2$, so that its Hodge dual $k = *f$ is the second rank conformal KY tensor obeying the equations

$$k_{\alpha\beta;\gamma} + k_{\gamma\beta;\alpha} = \frac{2}{D-1} (g_{\alpha\gamma} k^{\sigma}_{\beta;\sigma} + g_{\beta(\alpha} k_{\gamma)}^{\sigma}_{;\sigma}), \quad (12)$$

$$k_{[\alpha\beta,\gamma]} = 0. \quad (13)$$

The relation (13) implies that, at least locally, there exists a one-form (potential) b so that $k = db$.

We use the following ansatz for the conformal KY potential b for the MP metric (3)

$$2b = (r^2 + \sum_{i=1}^n a_i^2 \mu_i^2) dt + \sum_{i=1}^n a_i \mu_i^2 (r^2 + a_i^2) d\phi_i. \quad (14)$$

The corresponding conformal KY tensor k reads

$$k = \sum_{i=1}^n a_i \mu_i d\mu_i \wedge [a_i dt + (r^2 + a_i^2) d\phi_i] + r dr \wedge (dt + \sum_{i=1}^n a_i \mu_i^2 d\phi_i). \quad (15)$$

We emphasize that here and later on in similar formulas the summation over i is taken from 1 to n for both—even and odd number of spacetime dimensions D ; the coordinates μ_i are independent when D is even whereas they obey the constraint (5) when D is odd.

To prove that $k_{\mu\nu}$ obeys (12) it is convenient to use the Kerr–Schild form of the MP metrics. The required calculations are straightforward but rather long. The details of the proof can be found in [25]. For $D \leq 8$ we also checked directly the validity of the equation (12) by using the GRTensor program.

The Hodge dual of k ,

$$f = -*k, \quad (16)$$

is the KY tensor. We shall give the explicit expressions for f in the dimensions 4 and 5.

For $D = 4$ (the Kerr geometry) there is only one rotation parameter which, as usual, we denote by a . We also put $\mu_1 = \sin \theta$ and $\phi_1 = \phi$. In these notations one recovers the standard form of the Kerr metric and

$$f^{(4)} = r \sin \theta d\theta \wedge [adt + (r^2 + a^2) d\phi] - a \cos \theta dr \wedge (dt + a \sin^2 \theta d\phi). \quad (17)$$

This expression coincides with the KY tensor discovered by Penrose and Floyd [8].

For $D = 5$ there are 2 rotation parameters, a_1 and a_2 . Using the constraint (5) we write $\mu_2 = \sqrt{1 - \mu_1^2}$. Thus

we have

$$\begin{aligned}
-f^{(5)} = & rdt \wedge dr \wedge [a_2 \mu_1^2 d\phi_1 + a_1(1 - \mu_1^2) d\phi_2] \\
& + a_2 \mu_1 (r^2 + a_1^2) dt \wedge d\mu_1 \wedge d\phi_1 \\
& - a_1 \mu_1 (r^2 + a_2^2) dt \wedge d\mu_1 \wedge d\phi_2 \\
& + \mu_1 (r^2 + a_1^2)(r^2 + a_2^2) d\phi_1 \wedge d\phi_2 \wedge d\mu_1 \\
& + r\mu_1^2(1 - \mu_1^2)(a_2^2 - a_1^2) d\phi_1 \wedge d\phi_2 \wedge dr. \quad (18)
\end{aligned}$$

$$\begin{aligned}
-f^{(5)} = & rdt \wedge dr \wedge [a_2 \mu_1^2 d\phi_1 + a_1(1 - \mu_1^2) d\phi_2] \\
& + \mu_1 dt \wedge d\mu_1 \wedge [a_2(r^2 + a_1^2) d\phi_1 - a_1(r^2 + a_2^2) d\phi_2] \\
& + \mu_1 (r^2 + a_1^2)(r^2 + a_2^2) d\phi_1 \wedge d\phi_2 \wedge d\mu_1 \\
& + r\mu_1^2(1 - \mu_1^2)(a_2^2 - a_1^2) d\phi_1 \wedge d\phi_2 \wedge dr. \quad (19)
\end{aligned}$$

Using (15) and (16) one can easily obtain f in an explicit form for an arbitrary number of dimensions. However, the rank of the form f grows with the number of dimensions and the corresponding expressions become quite long.

It is easy to check that the following object constructed from the KY tensor,

$$K_{\mu\nu} = \frac{1}{(D-3)!} f_{\mu\alpha_1 \dots \alpha_{D-3}} f_{\nu}^{\alpha_1 \dots \alpha_{D-3}}, \quad (20)$$

is the second rank Killing tensor. Using (10) one can express $K_{\mu\nu}$ in terms of k

$$K_{\mu\nu} = k_{\mu\alpha} k_{\nu}^{\alpha} - \frac{1}{2} g_{\mu\nu} k_{\alpha\beta} k^{\alpha\beta}. \quad (21)$$

Evidently, $Q_{\mu\nu} = k_{\mu\alpha} k_{\nu}^{\alpha}$, is the conformal Killing tensor, satisfying, $Q_{(\alpha\beta;\gamma)} = g_{\alpha\beta} Q_{\gamma}$, where

$$Q_{\gamma} = \frac{1}{D+2} (2Q^{\kappa}_{\gamma;\kappa} + Q^{\kappa}_{\kappa;\gamma}). \quad (22)$$

The calculations give the following expression for the Killing tensor

$$\begin{aligned}
K^{\mu\nu} = & \sum_{i=1}^n \left[a_i^2 (\mu_i^2 - 1) g^{\mu\nu} + a_i^2 \mu_i^2 \delta_t^{\mu} \delta_t^{\nu} + \frac{1}{\mu_i^2} \delta_{\phi_i}^{\mu} \delta_{\phi_i}^{\nu} \right] \\
& + \sum_{i=1}^{n-1+\varepsilon} \delta_{\mu_i}^{\mu} \delta_{\mu_i}^{\nu} - 2Z^{(\mu} Z^{\nu)} - 2\xi^{(\mu} \zeta^{\nu)}, \quad (23)
\end{aligned}$$

where

$$\xi = \partial_t, \quad \zeta = \sum_{i=1}^n a_i \partial_{\phi_i}, \quad Z = \sum_{i=1}^{n-1+\varepsilon} \mu_i \partial_{\mu_i}. \quad (24)$$

The term constructed from the Killing vectors $\xi^{(\mu} \zeta^{\nu)}$ can be excluded from K . In the 4-dimensional spacetime (23) reduces to the Killing tensor obtained by Carter [4], while in the 5-dimensional case it coincides with the Killing tensor obtained in [16, 17] after the term $\xi^{(\mu} \zeta^{\nu)}$ is omitted.

The constructed KY and Killing tensors have direct connections with the isometries of the background MP geometry. First of all, for a second rank conformal KY tensor k in a Ricci-flat spacetime

$$\xi^{\mu} = \frac{1}{D-1} k^{\sigma\mu}_{;\sigma} \quad (25)$$

is a Killing vector [27]. In particular, for the MP metrics $\xi = \partial_t$. It is also easy to show that the vector $\eta_{\mu} = K_{\mu\nu} \xi^{\nu}$ possesses the property

$$\eta_{(\mu;\nu)} = -\frac{1}{2} \mathcal{L}_{\xi} K_{\mu\nu}. \quad (26)$$

For the MP metrics the Lie derivative of $\mathcal{L}_{\xi} K$ vanishes, so that η is a Killing vector. Calculations give

$$\eta = \sum_{i=1}^n a_i^2 \partial_t - \zeta. \quad (27)$$

The described ‘hidden’ symmetry of a higher dimensional rotating black hole implies the existence of an additional integral of motion. For example, for a freely moving particle with the velocity u^{μ} the Killing tensor (23) implies that the quantity $K_{\mu\nu} u^{\mu} u^{\nu}$ is constant. Because of the presence of the KY tensor f the classical spinning particles in the MP metric possesses enhanced worldline supersymmetry [24]. Similar symmetries are also valid on the quantum level. In particular, the operator $\nabla_{\mu} (K^{\mu\nu} \nabla_{\nu})$ commutes with the scalar Laplacian $\square = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu}$ [10, 26]. Using the KY tensor it is possible to construct an operator which commutes with the Dirac operator [10, 24, 29]. In general there exist deep relations between KY tensors and the supersymmetry [28].

It should be emphasized that for $D \geq 6$ the obtained KY and Killing tensors for the MP solutions do not guarantee the separation of variable in the Hamilton–Jacobi and Klein–Gordon equations. This can be easily illustrated for a particle. The spacetime symmetry of the $D = 2n + 1 + \varepsilon$ dimensional MP metric guarantees the existence of $n + 1$ integrals of motion. The normalization of the velocity gives one more integral. In order to have the separability, there must exist at least $J = D - (n + 2) = n + \varepsilon - 1$ additional integrals of motion. For $D = 4$ and $D = 5$ one has $J = 1$, thus the Killing tensor which exists in the MP metrics is sufficient for the separation of variables. For $D \geq 6$, $J > 1$, so that one cannot expect the separation of variables unless there exist additional ‘hidden’ symmetries. General conditions of the separability of the Hamilton–Jacobi and Klein–Gordon equations were obtained in [14]. Whether such additional symmetries exist in the MP metrics is an interesting open question.

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